Lectures on Massive Stars *Series 2*
Stellar Winds in Massive Stars

Fundamentals of Radiative Transfer in Expanding Media

Joris Josiek
ZAH/ARI, University of Heidelberg

supervised by
Jo Puls (LMU Munich)

UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

17 April 2025

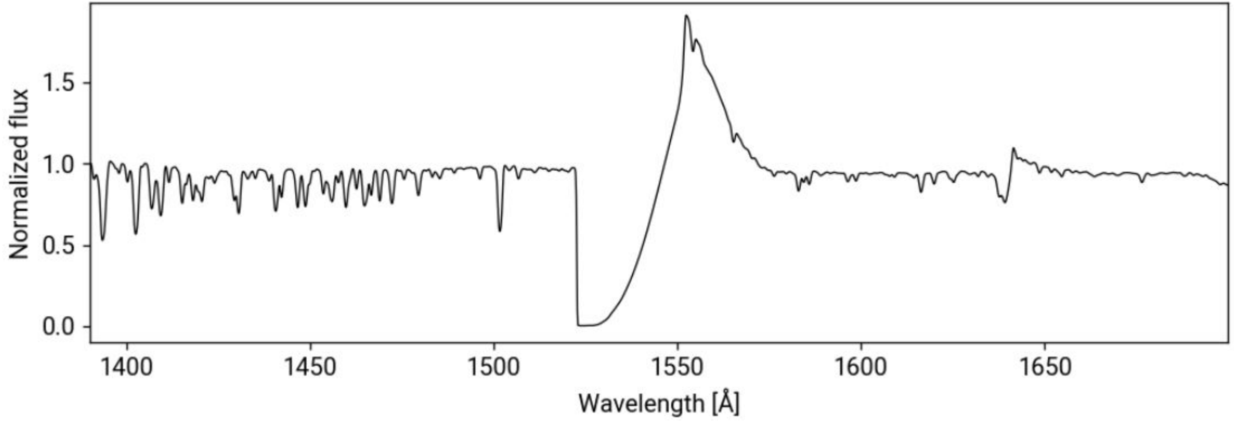
Radiation in hot stars interacts with matter in complex ways.

Need to understand radiative transfer in order to study:

1 Emergent spectra

2 Wind driving

→ **Analysis of stellar atmospheres**



Synthetic spectrum of a WNh star computed with PoWR

Radiative transfer – the basics

Specific intensity

$$\mathrm{d}E = I_\nu(\mathbf{r}, \mathbf{n}, t)\, \mathrm{d}A\, \mathrm{d}\Omega\, \mathrm{d}t\, \mathrm{d}\nu$$

- Energy transported by radiation per (projected) **area**, per **solid angle**, per **time**, per **frequency.**
- Can generally depend on position, direction, time and frequency.

Unlike flux, specific intensity does not dilute over distance.

→ Only affected by interaction with matter → Radiative Transfer Equation

Radiative Transfer Equation (RTE)

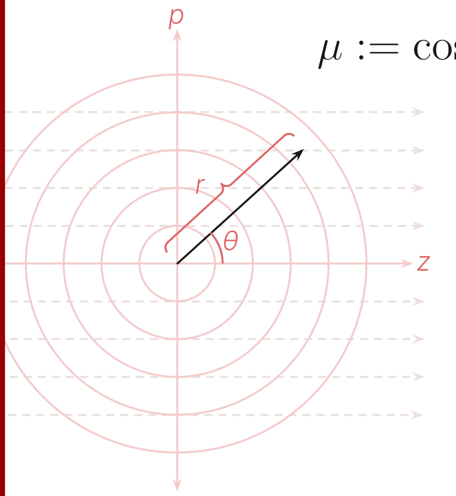
$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{n} \cdot \nabla\right) I_\nu(\mathbf{r}, \mathbf{n}, t) = \eta_\nu(\mathbf{r}, \mathbf{n}, t) - \chi_\nu(\mathbf{r}, \mathbf{n}, t) I_\nu(\mathbf{r}, \mathbf{n}, t)$$

change in intensity = emission — absorption

Analytical solutions only for very special cases

Integrating full RTE is numerically expensive; accuracy depends on method

Spherical symmetry



$$\mu := \cos\theta$$

$$\mathbf{r} = (p, z) = (r\sin\theta, r\cos\theta) = \left(r\sqrt{1-\mu^2}, r\mu\right)$$

Propagation of light: $\mathbf{n} := \hat{\mathbf{z}}$

$$\left(\mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\right) I_\nu(r,\mu) = \eta_\nu(r,\mu) - \chi_\nu(r,\mu) I_\nu(r,\mu)$$

Stationary RTE in spherical geometry

$$\frac{\mathrm{d}I_\nu(p,z)}{\mathrm{d}z} = \eta_\nu(p,z) - \chi_\nu(p,z) I_\nu(p,z)$$

Stationary RTE in p-z geometry

Common definitions

Optical depth

$$\tau_\nu(p, z) = \int_z^\infty \chi_\nu(p, z')\,\mathrm{d}z'$$

Source function

$$S_\nu := \frac{\eta_\nu}{\chi_\nu}$$

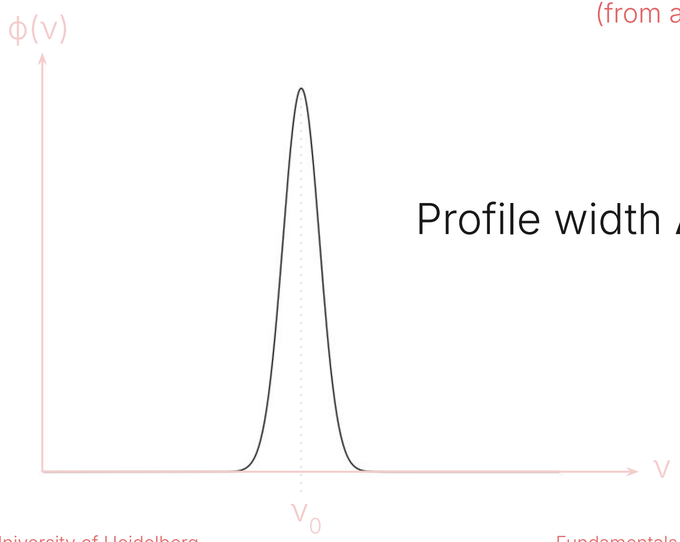
Profile function

Opacity for one spectral line

$$\chi_\nu(\mathbf{r}) = \bar{\chi}(\mathbf{r})\phi(\nu, \mathbf{r})$$

opacity — total line opacity (from atomic data) — profile function

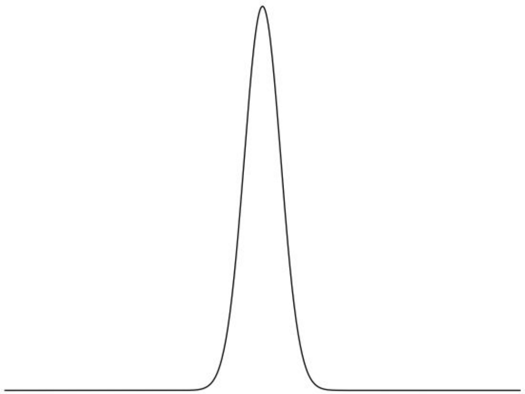
$$\bar{\chi} = \int \chi_\nu \, \mathrm{d}\nu \qquad \int \phi(\nu) \, \mathrm{d}\nu = 1$$



Profile width Δν due to thermal motion (and random motion)

$$\Delta\nu = \nu_0 \frac{v_{\mathrm{th}}}{c}$$

Intrinsic line profile

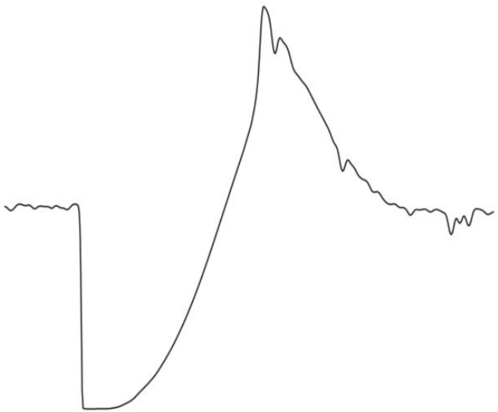


$\Delta v \sim v_{th}$ ~ 10 km/s

?

→

Emergent line profile



$\Delta v \sim v_{wind}$ ~ 1000 km/s

Formal integral

Integrating the RTE yields the "formal solution" for $I_\nu(\mathbf{r})$

$$I_\nu(\tau_\nu) = I_\nu(\tau_0) e^{-(\tau_0 - \tau_\nu)} + \int_{\tau_\nu}^{\tau_0} S(\tau'_\nu)\, e^{-(\tau'_\nu - \tau_\nu)}\, \mathrm{d}\tau'_\nu$$

Problems with numerical integration:

- Unknown source function?
- Highly resolved grids in space and frequency required (narrow lines vs. fast winds)
- Opacities and emissivities are non isotropic (due to Doppler-shift), and depend on r and μ.

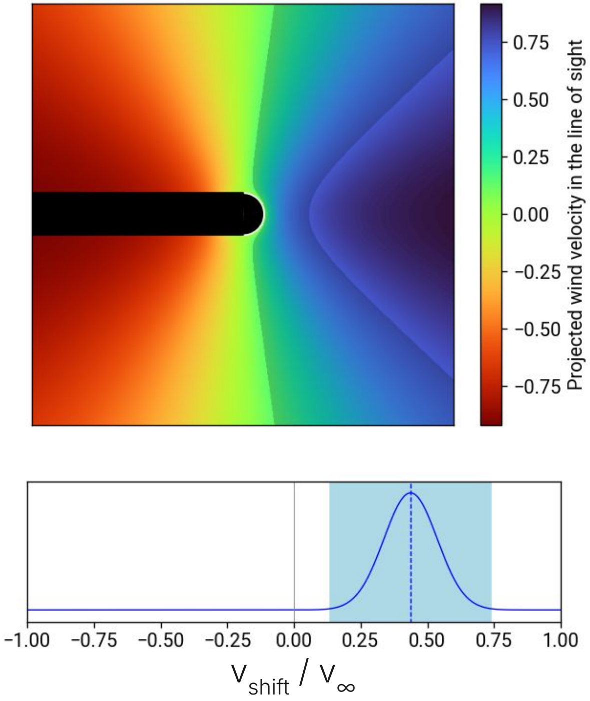
→ Large errors when resolution not sufficient

Sobolev Theory

Sobolev (1960)
Rybicki & Hummer (1978)
Puls, Canary Winter-School (2017)

The Resonance Zone

* This figure was an animated gif in the original presentation.



Consider a spectral line with restframe wavelength ν_0

A photon with restframe wavelength ν can only be absorbed in the region where the line center is Doppler-shifted to within a few thermal widths of the photon.
→ **Resonance Zone**

Radial velocity projected in the line of sight

$$\left| \nu - \nu_0 \left(1 + \frac{\mu v(r)}{c} \right) \right| \lesssim 3\nu_0 \frac{v_{\text{th}}}{c}$$

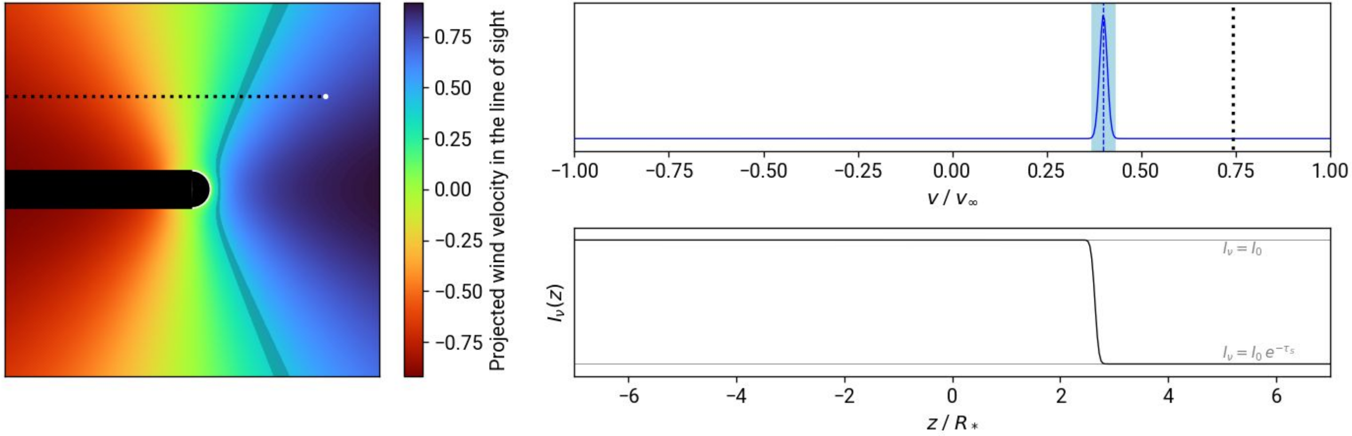
photon frequency

shifted central line frequency

3x thermal line width

The Resonance Zone

* This figure was an animated gif in the original presentation.



$v_{\text{th}} \ll v_\infty$

Assume the resonance zone is “narrow”

→ macrovariables (opacity, source function, v gradient) are (almost) constant

→ there is no interaction in most of the medium (if continuum is weak)

The Sobolev optical depth

$$\tau_S(\nu, p) = \int \bar{\chi}(p, z)\, \phi(\nu_{\mathrm{CMF}}(\nu, p, z))\, \mathrm{d}z$$

Observed photon frequency (ν); Impact parameter (p); Line opacity ($\bar{\chi}$); Profile function (ϕ); Photon frequency in the comoving frame (ν_{CMF}); Integrate over photon path ($\mathrm{d}z$)

$$\nu_{\mathrm{CMF}}(\nu, r, \mu) = \nu \left(1 - \frac{\mu v(r)}{c}\right)$$

Position in polar (r,µ) coordinates (r, μ); Local Doppler-shift ($\left(1 - \frac{\mu v(r)}{c}\right)$)

Transform integration variable from $\mathrm{d}z$ to $\mathrm{d}\nu_{\mathrm{CMF}}$:

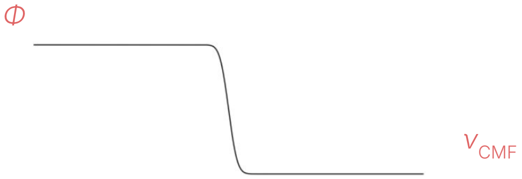
$$\frac{\mathrm{d}\nu_{\mathrm{CMF}}}{\mathrm{d}z} = -\frac{\nu}{c}\left(\mu^2 \frac{\mathrm{d}v}{\mathrm{d}r} + (1 - \mu^2)\frac{v}{r}\right)$$

Geometry of coordinate systems

$$r(p, z) = \sqrt{p^2 + z^2}$$

$$\mu(p, z) = \frac{z}{\sqrt{p^2 + z^2}}$$

The Sobolev optical depth



Integrated line profile (from blue to red)

$$\tau_S(\nu, p) = \int \frac{\bar{\chi}(p,z)}{\frac{\nu}{c}\left|\mu^2 \frac{\mathrm{d}v}{\mathrm{d}r} + (1-\mu^2)\frac{v}{r}\right|} \phi(\nu_{\mathrm{CMF}}) \mathrm{d}\nu_{\mathrm{CMF}} = \left.\frac{\bar{\chi}(p,z)}{\frac{\nu}{c}\left|\mu^2 \frac{\mathrm{d}v}{\mathrm{d}r} + (1-\mu^2)\frac{v}{r}\right|}\right|_{\mathrm{RZ}(\nu,p)} \int \phi(\nu_{\mathrm{CMF}}) \mathrm{d}\nu_{\mathrm{CMF}}$$

Sobolev approximation: **macrovariables are constant in the resonance zone!**
(including dv/dr)

$$\tau_S(\nu, p) = \left.\frac{\bar{\chi}}{\frac{\nu}{c}\left|\mu^2 \frac{\mathrm{d}v}{\mathrm{d}r} + (1-\mu^2)\frac{v}{r}\right|}\right|_{\mathrm{RZ}(\nu,p)}$$

Evaluate in the resonance zone
(for each *v* and *p*)

Formal solution with Sobolev approximation

For each ν and p: ① Find the location of the RZ,
② Compute the Sobolev optical depth.

Then:

Before passing the RZ:

$$I_\nu(p, z) = I_\nu(p, z_{\mathrm{back}})$$

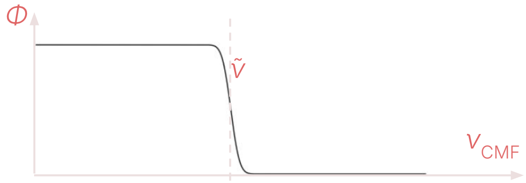
After passing the RZ:

$$I_\nu(p, z) = I_\nu(p, z_{\mathrm{back}})\, e^{-\tau_S(\mathrm{RZ})} + S_{\mathrm{RZ}}(1 - e^{-\tau_S(\mathrm{RZ})})$$

General:

$$I_\nu(p, z) = I_\nu(p, z_{\mathrm{back}})\, e^{-\tau_S(\mathrm{RZ})\Phi(\nu_{\mathrm{CMF}})} + S_{\mathrm{RZ}}(1 - e^{-\tau_S(\mathrm{RZ})\Phi(\nu_{\mathrm{CMF}})})$$

Integrated line profile



The specific intensity is calculated <u>locally</u>!

Profile-weighted mean intensity

From previous slide: $I_\nu(p,z) = I_\nu(p, z_{\text{back}})\, e^{-\tau_S(\text{RZ})\Phi(\nu_{\text{CMF}})} + S_{\text{RZ}}(1 - e^{-\tau_S(\text{RZ})\Phi(\nu_{\text{CMF}})})$

$$\bar{I}(p,z) = \int I_\nu(p,z)\,\phi(\nu_{\text{CMF}})\,\mathrm{d}\nu_{\text{CMF}}$$

Needed for rate equations and computing radiative acceleration.

Only photons with a comoving frequency close to the line center contribute!

Change coordinate system ($p,z \rightarrow r,\mu$) and integrate over $\mathrm{d}\Phi = -\phi(\nu_{\text{CMF}})\,\mathrm{d}\nu_{\text{CMF}}$

Profile-weighted mean intensity

$$\bar{I}(r,\mu) = I_{\text{back}}(p)\frac{1-e^{-\tau_S(r,\mu)}}{\tau_S(r,\mu)} + S(r)\left(1 - \frac{1-e^{-\tau_S(r,\mu)}}{\tau_S(r,\mu)}\right)$$

This is completely **local**
(i.e. no information is needed from any other part of the star)

Optically thick lines: $\bar{I}(\mathbf{r}) = S(\mathbf{r})$

Optically thin lines: $\bar{I}(\mathbf{r}) = I_{\text{back}}(p)$

from 1st-order expansion of τ_S

Mean intensity and Eddington flux

$$\bar{I}(r,\mu) = I_{\text{back}}(p)\frac{1-e^{-\tau_S(r,\mu)}}{\tau_S(r,\mu)} + S(r)^{***}\left(1-\frac{1-e^{-\tau_S(r,\mu)}}{\tau_S(r,\mu)}\right)$$

*** S is mostly independent of μ since the angle-dependence is in the profile function which is usually the same for emissivity and opacity and therefore cancels out when considering $S = \eta/\chi$.

$$\bar{J}(r) = \frac{1}{2}\int \bar{I}(r,\mu)\,\mathrm{d}\mu$$

→ rate equations

$$\bar{H}(r) = \frac{1}{2}\int \bar{I}(r,\mu)\,\mu\,\mathrm{d}\mu$$

→ radiative acceleration

Independent of source function in the Sobolev approximation (since only μ^2 plays a role and thus the second term in $\bar{I}$ is fore-aft symmetrical)

When is the Sobolev approximation valid?

When macroscopic variables are constant in the RZ.

Scale-height of variable X $\left\{ \frac{X}{\mathrm{d}X/\mathrm{d}r} \gg \frac{v_{\mathrm{th}}}{\mathrm{d}v/\mathrm{d}r} \right\}$ Sobolev length = width of the RZ

This is the case in fast-wind regimes: winds above the thermal point and SN remnants

It is <u>not</u> appropriate for lines formed below the sonic point, where $v \lesssim v_{\mathrm{th}}$

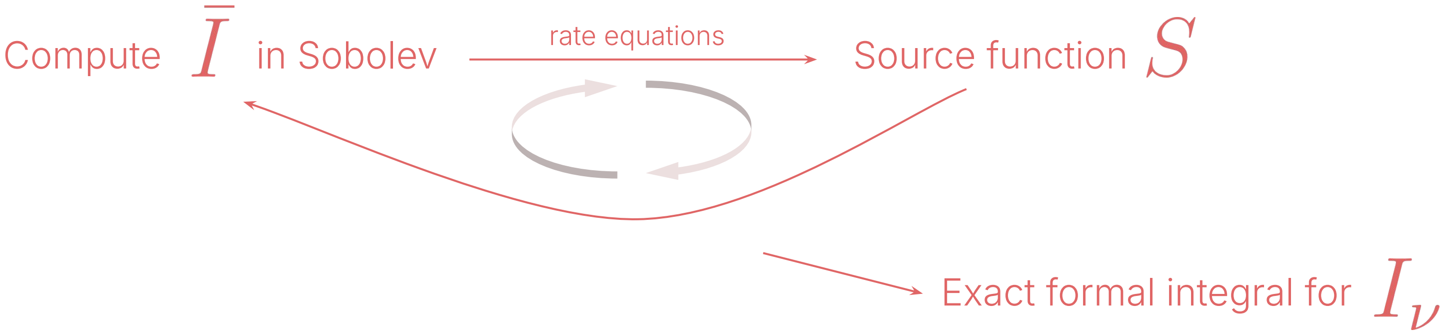
(regions where v=0 would be inside the RZ, which then includes the entire stellar interior)

Also <u>not</u> appropriate for regions with high v curvature (where dv/dr ≠ const.)

e.g. at the sonic point

Extensions of Sobolev theory

- Continuum   Hummer & Rybicki (1985)
- Gradients of S Puls & Hummer (1988)
- Multiple lines   Puls (1987)
- "Sobolev with Exact Integration" (SEI)   Hamann (1981), Lamers et al. (1987)



The comoving frame (CMF) method

Lucy (1971),
Mihalas et al. (1975),
Hamann (1985)

Radiative transfer equation:

$$\pm\frac{\mathrm{d}I^{\pm}}{\mathrm{d}z} = \eta\left(r, \nu\left(1-\frac{\mu v}{c}\right)\right) - \chi\left(r, \nu\left(1-\frac{\mu v}{c}\right)\right) I^{\pm}$$

Simple transformation into CMF with $\nu_{\mathrm{CMF}} = \nu\left(1-\frac{\mu v}{c}\right)$ and $\left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{\nu} = \left.\frac{\partial}{\partial z}\right|_{\nu_{\mathrm{CMF}}} + \left.\frac{\partial \nu_{\mathrm{CMF}}}{\partial z}\right|_{\nu} \left.\frac{\partial}{\partial \nu_{\mathrm{CMF}}}\right|_{z}$

geometrical factor $Q = \left|\mu^2\frac{\mathrm{d}v}{\mathrm{d}r} + (1-\mu^2)\frac{v}{r}\right|$

$$\pm\frac{\partial I^{\pm}}{\partial z} - \frac{\nu_{\mathrm{CMF}} Q}{c}\frac{\partial I^{\pm}}{\partial \nu_{\mathrm{CMF}}} = \eta(r, \nu_{\mathrm{CMF}}) - \chi(r, \nu_{\mathrm{CMF}}) I^{\pm}$$

(v ≪ c)

The comoving frame (CMF) method

$$\pm\frac{\partial I^{\pm}}{\partial z} - \frac{\nu_{\mathrm{CMF}} Q}{c}\frac{\partial I^{\pm}}{\partial \nu_{\mathrm{CMF}}} = \eta(r, \nu_{\mathrm{CMF}}) - \chi(r, \nu_{\mathrm{CMF}}) I^{\pm}$$

Numerical integration schemes: *implicit* (Mihalas+1975) or *semi-implicit* (Hamann 1981)

ADVANTAGES

- Only a small range of v_{CMF} around the line center needs to be considered
- η and χ are **isotropic** in the CMF
- $\bar{J}$ and $\bar{H}$ don't need to be transformed into the observer's frame

POTENTIAL ISSUES

- Only covers the non-relativistic limit $v \ll c$
- Boundary conditions in space and initial conditions in frequency required.

From this equation, the Sobolev approximation can be exactly obtained by neglecting $\frac{\partial}{\partial z}$–term.

Sobolev vs. CMF

Emergent line profile

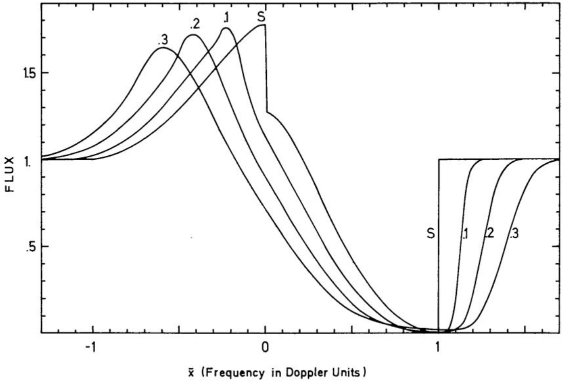
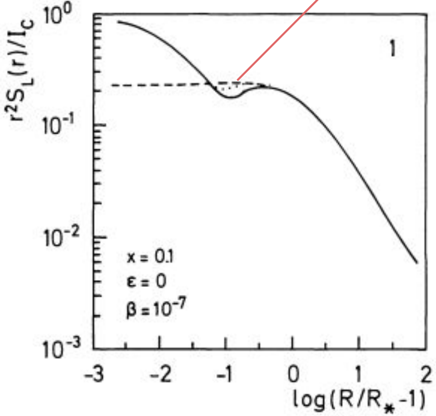


Fig. 2. Emergent flux profiles for the parameters $\kappa_0=10$, $\alpha=0$, $\beta=1/2$. The Sobolev result is labelled "S", the comoving-frame results by their parameter $v_D/v_\infty=0.1$, 0.2 or 0.3, respectively

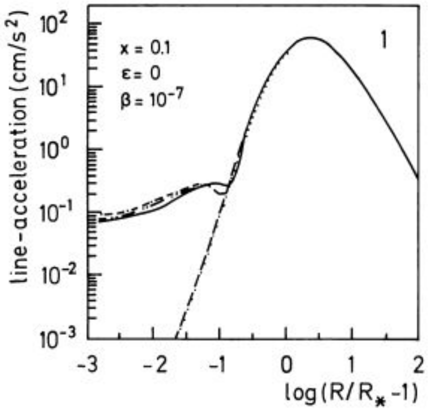
Hamann (1981)

source function dip

Source function



Radiative acceleration



Puls & Hummer (1988)

—— CMF

- - - - Sobolev

······ Sobolev with Continuum

Conclusions

Sobolev theory allows for fast solution of radiative transfer for wind lines (above the thermal/sonic point or if dv/dr = const.)

> *Concept of the "Resonance Zone" reduces integrals to a local calculation.*

→ Particularly powerful when including the extensions (continuum, multi-line, etc...)

For more general applications (e.g. quasi-photospheric lines), CMF is used.